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Chiral bosonization for non-commutative fields

Ashok Das

Department of Physics and Astronomy, University of Rochester
Rochester, NY 14627-0171, U.S.A.
E-mail: das@pas.rochester.edu

Jorge Gamboa

Departamento de Física, Universidad de Santiago de Chile
Casilla 307, Santiago 2, Chile
E-mail: jgamboa@lauca.usach.cl

Fernando Méndez

INFN, Laboratorio Nazionali del Gran Sasso
SS, 17bis, 67010 Asergi (L'Aquila), Italy
E-mail: fernando.mendez@lngs.infn.it

Justo López-Sarrión

Departamento de Física Teórica, Universidad de Zaragoza
Zaragoza 50009, Spain
E-mail: justo@dftuz.unizar.es

Abstract: A model of chiral bosons on a non-commutative field space is constructed and new generalized bosonization (fermionization) rules for these fields are given. The conformal structure of the theory is characterized by a level of the Kac-Moody algebra equal to $(1+\theta^2)$ where θ is the non-commutativity parameter and chiral bosons living in a non-commutative fields space are described by a rational conformal field theory with the central charge of the Virasoro algebra equal to 1. The non-commutative chiral bosons are shown to correspond to a free fermion moving with a speed equal to $c' = c\sqrt{1+\theta^2}$ where c is the speed of light. Lorentz invariance remains intact if c is rescaled by $c \to c'$. The dispersion relation for bosons and fermions, in this case, is given by $\omega = c'|k|$.

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1. Introduction

Bosonization is an important theoretical technique in many fields such as string theory [1], condensed matter physics [2] and mathematical physics [3] (for detailed references on bosonization see [4]). From a purely physical point of view, bosonization is a duality relation between the strong and the weak coupling limits and, therefore, it can provide information about the nonperturbative sector of quantum field theories.

It seems quite natural to consider the present relativistic quantum field theories as effective descriptions where for energies over 10 TeV and more, Lorentz invariance should be replaced by another one -up to now unknown symmetry [5, 6, 7]. Several recent events violating the GZK bound of ultra-high energy cosmic rays support this possibility [8] and a kinematical analysis suggests that for momenta over 10^{16} eV, new non-commutative effects could take place and, as a consequence, a tiny violation of the micro-causality principle would be possible [9]. The non-commutative theories, of course, present examples where Lorentz invariance is violated. In a different context, Coleman and Glashow [10] have also discussed about such a possibility in particle physics processes. However, in the absence of Lorentz invariance, it is not clear *a priori* how to determine the correct dispersion relations for bosons and fermions.[1]

[1]For a systematic discussion about possible violations of Lorentz invariance see [11].

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In this sense, an analysis in a $1+1$ dimensional system can be useful in analyzing this problem. The goal of the present paper is to analyze the abelian bosonization for a two dimensional system and to show how the bosonization formulae are modified in the non-commutative space of fields. More precisely, we will show below that for chiral bosons in a non-commutative field space conformal invariance continues to hold and that the non-commutativity in the field space leads to 'free' fermions when chiral bosons are fermionized. The relativistic symmetry remains intact provided the speed of light is rescaled by a factor depending on the parameter of non-commutativity. This last fact can be seen as a transition between the commutative and the non-commutative limits.

The paper is organized as follows. In section 2 we review very briefly the analysis of chiral bosons in the commutative field space. In section 3 we deform the canonical Dirac algebra (commutation relations) and analyze some of the basic features of such a system. In section 4 we generalize the formulae for bosonization (fermionization) for such a theory. The relativistic invariance of such a theory is discussed in section 5 and we conclude with a brief summary in section 6. In appendix A, we canonically quantize this theory and derive the time ordered Green's function for such a theory. In appendix B, we discuss briefly the modified Lorentz transformations in this theory.

2. Chiral bosons in commutative space of fields

In this section, we will very briefly recapitulate the essential features of quantization of a chiral boson in the commutative field space before going into chiral bosons in a non-commutative field space. Let us consider a massless scalar field in two dimensional Minkowski space-time. In such a case, the field ϕ can be decomposed as

$$\phi = \phi_L + \phi_R \,, \tag{2.1}$$

where the $\phi_L(\phi_R)$ describes a left (right) moving component. The left and right moving fields are also represented as $\phi_\pm$ (denoting the dependence on the coordinates $x \pm ct$) which we will use in our discussions. These individual components of the scalar field are known as chiral bosons.

In the local description, the dynamics of left and right handed chiral bosons can be obtained from the action

$$S_a = \int d^2x \left[a\phi'_a \dot{\phi}_a - (\phi'_a)^2 \right] , \tag{2.2}$$

where the signs $a = \pm$ represent respectively the left and the right movers [12] and there is no summation over a. Here the prime and the dot denote respectively a derivative with respect to x and t.

The action in (2.2) is constrained and the hamiltonian analysis leads to the second class constraint

$$\chi_a = \pi_a - a\phi'_a \,, \qquad a = \pm \,, \tag{2.3}$$

whose Poisson bracket (with the usual canonical relations) yields

$$[\chi_a(x), \chi_b(y)] = -2a\delta_{ab}\delta'(x-y) \,. \tag{2.4}$$

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With (2.3)–(2.4), the Dirac brackets for the field variables can be easily calculated and they take the forms

$$[\phi_a(x), \phi_b(y)]_D = -\frac{a}{2}\delta_{ab}\epsilon(x-y)\,, \tag{2.5}$$

$$[\phi_a(x), \pi_b(y)]_D = \frac{1}{2}\delta_{ab}\delta(x-y)\,, \tag{2.6}$$

$$[\pi_a(x), \pi_b(y)]_D = \frac{a}{2}\delta_{ab}\delta'(x-y)\,, \qquad a,b=\pm\,, \tag{2.7}$$

where $\epsilon(x-y)$ represents the alternating step function. We note from (2.5)–(2.7) that the left and the right moving components define independent degrees of freedom and have decoupled Dirac brackets.

As is clear from (2.2), the left and the right movers are governed respectively by the hamiltonians

$$H_a = \int dx\,(\phi_a')^2\,, \qquad a = \pm(\text{no summation})\,, \tag{2.8}$$

and the hamiltonian equations of motion take the forms

$$\begin{aligned} \dot{\phi}_a &= a\phi_a'\,, \\ \dot{\pi}_a &= a\phi_a'\,, \end{aligned} \tag{2.9}$$

or equivalently, the second order equation (which can also be seen from the action in (2.2))

$$\dot{\phi}_a' = a\phi_a''\,. \tag{2.10}$$

The canonical quantization of this system can now be carried out in a straightforward manner by replacing the Dirac brackets by commutators. We will not go into the details of this analysis which has been discussed extensively in the literature, see e.g. [13].

3. Deforming the commutators

We will next study the chiral boson in a non-commutative field space.[2] To that end, we deform the Dirac brackets of (2.5)–(2.7) and define the basic commutators (for chiral bosons in a non-commutative field space)

$$[\phi_a(x), \phi_b(y)] = i\Delta_{ab}\epsilon(x-y)\,, \tag{3.1}$$

where

$$\Delta_{ab} = \frac{a}{2}\left[\epsilon_{ab}\theta - \delta_{ab}\right], \tag{3.2}$$

and θ is the non-commutative parameter in the field space. The commutator in (3.1) is manifestly anti-symmetric and satisfies Jacobi identity. Furthermore, in the limit $\theta = 0$, it reduces to the earlier commutation relations. We note that unlike (2.7), the left and right moving fields now do not commute and they become coupled. Indeed, as (3.1) and (3.2)

[2]A related problem to our research is, of course, string theory on a B-field background see [14]. This last problem is very similar to the Landau problem in quantum mechanics.

mix left and right movers it is quite natural to assume that the dynamics is described by the following hamiltonian operator

$$H = \sum_a \int dx : (\phi'_a)^2 : , \tag{3.3}$$

with appropriate normal ordering. The operator (3.3) describes free left and right movers for the theory defined in a commutative field space. In the present case, however, it describes an interacting theory because of the deformed commutation relations.[3]

This last feature is very similar to the Landau problem in quantum mechanics, where the interacting theory is described by a 'free' hamiltonian

$$H = \frac{1}{2}p^2 ,$$

with the magnetic interaction included in the commutator, i.e.

$$[p_i, p_j] = i\epsilon_{ij} B ,$$

where B is the constant magnetic field.

The Heisenberg equations following from (3.3) are

$$\dot{\phi}_a = 2 \sum_{b=\pm} \Delta_{ab} \phi'_b . \tag{3.4}$$

In light-cone coordinates, these equations take the explicit forms

$$\begin{aligned} \partial_- \phi_+ &= -\theta \phi'_- \\ \partial_+ \phi_- &= -\theta \phi'_+ . \end{aligned} \tag{3.5}$$

Equivalently, they can be written as

$$\begin{aligned} \Box \phi_+ &= \theta^2 \partial_x^2 \phi_+ \\ \Box \phi_- &= \theta^2 \partial_x^2 \phi_- , \end{aligned} \tag{3.6}$$

and, therefore, the effect of the non-commutativity in the field space appears to modify the speed of light

$$c \to c' = c\sqrt{1+\theta^2} . \tag{3.7}$$

Correspondingly, the dispersion relation, in this case, becomes

$$E = \pm c \sqrt{1+\theta^2} |k| = \pm c' |k| . \tag{3.8}$$

We note that the matrix Δ_{ab} in (3.2) is invertible with an inverse given by

$$\Delta^{-1}_{ab} = \frac{4}{1+\theta^2} \Delta_{ab} . \tag{3.9}$$

[3] In the reference [15] was shown — for string theory on the B-background — that the open strings satisfy a mixed boundary condition, which basically mixes the left and right movers. In this sense our results are consistent with [15].

Consequently, the action for the system can also be written as

$$S=\int d^2x\left(-\frac{2}{1+\theta^2}\,\phi_a'\Delta_{ab}\dot{\phi}_b-(\phi_a')^2\right), \tag{3.10}$$

where summation over repeated indices is understood. The commutation relations in (3.1) can be easily checked to follow from this action. Furthermore, the dynamical equations for the chiral bosons (in non-commutative field space) now follow to be (which is consistent with (3.4))

$$\partial_x\begin{pmatrix}\partial_t-\partial_x & \theta\partial_x\\ \theta\partial_x & \partial_t+\partial_x\end{pmatrix}\begin{pmatrix}\phi_+\\ \phi_-\end{pmatrix}=0\,. \tag{3.11}$$

The Green's function $G\,(x-y)$ can now be defined to satisfy

$$\partial_x\begin{pmatrix}\partial_t-\partial_x & \theta\partial_x\\ \theta\partial_x & \partial_t+\partial_x\end{pmatrix}G\,(x-y)=\delta^2(x-y)\,, \tag{3.12}$$

with $G\,(x-y)$ a 2×2 matrix. The Green's function is easily determined in the momentum space, namely, defining

$$G(x)=\frac{1}{(2\pi)^2}\int d^2k\,\tilde{G}(k)e^{ik\cdot x}\,, \tag{3.13}$$

we obtain

$$\tilde{G}(k)=\frac{1}{k\,[(k^0)^2-(1+\theta^2)k^2]}\begin{pmatrix}k^0-k & \theta\,k\\ \theta\,k & k^0+k\end{pmatrix}. \tag{3.14}$$

The momentum integration can be carried out in a straightforward manner using the Feynman prescription. As we will show in appendix A, the time ordered Green's function of the theory $\mathcal{G}$ is related to G (with Feynman's prescription) through the relation

$$\mathcal{G}=iG\Delta(\theta)\,, \tag{3.15}$$

where $\Delta(\theta)$ has the matrix form

$$\Delta(\theta)=\frac{1}{2}\begin{pmatrix}-1 & \theta\\ \theta & 1\end{pmatrix}, \tag{3.16}$$

which can also be read out from the defining relation in (3.2) (the matrix is labelled by the indices $\pm$). In the coordinate space, the time ordered Green's function (as we will show in appendix A or as can be evaluated directly from (3.13), (3.14) and (3.15)) takes the form

$$\begin{aligned}\mathcal{G}_{\pm\pm}(x) &= \mp\frac{1}{8\pi}\left[\left(1\mp\frac{c'}{c}\right)\ln|x-c't|-\left(1\pm\frac{c'}{c}\right)\ln|x+c't|\right],\\ \mathcal{G}_{+-}(x) &= \frac{\theta}{8\pi}\ln\left|\frac{(x-c't)}{(x+c't)}\right|,\end{aligned} \tag{3.17}$$

where we have identified

$$c'=c\,\sqrt{1+\theta^2}\,, \tag{3.18}$$

with c representing the speed of light and we have omitted terms involving the massive regulator.

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These last relations provide a complete solution for chiral bosons living on non-commutative field space. In particular, the off-diagonal Green's functions in (3.17) reflect the effects due non-commutativity in the field space. We note that when $\theta = 0$, the Green's functions reduce to those of conventional chiral bosons where the left and the right movers are decoupled.

4. Fermionization of chiral bosons in a non-commutative field space

The fermionization of chiral bosons in a non-commutative field space is, in principle, straightforward [16]. In fact, let us define

$$\psi_+ = C \ : e^{i\alpha(\phi_+ - \theta\phi_-)} :, \tag{4.1}$$
$$\psi_- = C \ : e^{-i\alpha(\phi_- + \theta\phi_+)} :, \tag{4.2}$$

where C is a normalization constant (possibly infinite) and :: denotes normal ordering. Using (4.1) and (4.2) we can compute the current algebra following the standard formulae of conformal field theory [3, 17]. Indeed, using

$$: e^A :: e^B := e^{\langle AB \rangle} : e^A e^B :,$$

where A, B denote two arbitrary operators, we can determine the fermionic currents to be of the forms

$$\begin{aligned}
\psi_+^\dagger(x)\psi_+(y) &\sim e^{\alpha^2(1+\theta^2)\mathcal{G}_{++}(x-y)} : e^{i\alpha(\Delta\phi_+ - \theta\Delta\phi_- + \cdots)} :, \\
\psi_+^\dagger(x)\psi_-(y) &\sim e^{\alpha^2(1+\theta^2)\mathcal{G}_{+-}(x-y)} : e^{-\alpha(\phi_+ - \theta\phi_-)(x)} \, e^{-\alpha(\phi_- + \theta\phi_+)(y)} :, \\
\psi_-^\dagger(x)\psi_+(y) &\sim e^{\alpha^2(1+\theta^2)\mathcal{G}_{-+}(x-y)} : e^{-\alpha(\phi_- - \theta\phi_+)(x)} \, e^{-\alpha(\phi_+ + \theta\phi_-)(y)} :, \\
\psi_-^\dagger(x)\psi_-(y) &\sim e^{\alpha^2(1+\theta^2)\mathcal{G}_{--}(x-y)} \ : e^{i\alpha(\Delta\phi_- - \theta\Delta\phi_+ + \cdots)} : .
\end{aligned} \tag{4.3}$$

For convenience, the coefficient α can be chosen to be

$$\alpha = \frac{4\pi c^3}{(c')^3} = \frac{4\pi}{(1+\theta^2)^{3/2}}, \tag{4.4}$$

although this is not necessary.

We note from (3.17) that, in the limit $x \to y$, the Green's functions behave as $\ln|x-y|$ and the first and the last relations in (4.3) give (with the use of the forms of the Green's function in (3.17))

$$\psi_\pm^\dagger(x)\psi_\pm(y) = \alpha\,[\partial_x\phi_\pm - \theta\partial_x\phi_\mp + \cdots], \tag{4.5}$$

where $\cdots$ denote terms (possibly non-regular) that are not relevant for the calculation of the current algebra. However, we note that if we take off-diagonal combinations such as $\psi_-^\dagger\psi_+$, the coefficient in the exponent is positive and, therefore, in the limit $x \to y$,

$$\psi_-^\dagger(x)\psi_+(y) = 0 = \psi_+^\dagger(x)\psi_-(y)\,. \tag{4.6}$$

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Thus, we find that the bosonization formulae corrected by non-commutativity in the $\pm$ sector [18] take the forms

$$J_{\pm} = \frac{\alpha}{\sqrt{\pi}} \left[\pm\partial_{\pm}\phi_{\pm} + \theta\partial_{\mp}\phi_{\mp} \right] . \tag{4.7}$$

In such a case, therefore, we cannot have a mass term present in the hamiltonian and the model considered here including the deformed algebra (3.1) defines a conformal field theory.

The current algebra can now be calculated easily using the commutation relations in (3.1) and takes the form

$$\begin{aligned} [J_{\pm}(x), J_{\pm}(y)] &= \pm\frac{i}{2}k_1\ \delta'(x-y)\,, \\ [J_{+}(x), J_{-}(y)] &= \frac{i}{2}k_2\delta'(x-y)\,, \end{aligned} \tag{4.8}$$

where the levels in the algebra in (4.8) are given by

$$k_1 = 1\,, \qquad k_2 = \theta\,.$$

Equation (4.8) represents two coupled U(1) Kac-Moody algebras where the levels are different for the mixed and the unmixed commutators. However, this is not a problem since the algebra can be diagonalized by going to a different basis.

In order to do that, we note that although each coefficient in the fermionic current is different, one can posit the general hamiltonian operator

$$H = \int dx : \left[J_{+}^2 + J_{-}^2 + \beta J_{+}J_{-} \right] :, \tag{4.9}$$

where β is a constant. However, we note that the current algebra is diagonalized for $\beta = 0$. Therefore, we will take this value for β. We observe that if instead of the currents $J_{\pm}$ we consider the new set of currents $j_{\pm}$ defined as

$$\begin{aligned} j_{+} &= J_{+}\cosh\omega - J_{-}\sinh\omega\,, \\ j_{-} &= J_{+}\sinh\omega + J_{-}\cosh\omega\,, \end{aligned} \tag{4.10}$$

then, in this new basis, the current algebra has the form

$$[j_{\pm}(x), j_{\pm}(y)] = \pm\frac{i}{2}(1+\theta^2)\ \delta'(x-y) \tag{4.11}$$

$$[j_{-}(x), j_{+}(y)] = 0\,, \tag{4.12}$$

provided (this follows from (4.12))

$$\sinh 2\omega + \theta = 0\,.$$

It follows now from (4.11)-(4.12) that the new levels of the Kac-Moody algebra are

$$k_1 = 1 + \theta^2\,, \qquad k_2 = 0\,, \tag{4.13}$$

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and that the model of chiral bosons in a non-commutative field space is a conformal field theory. The hamiltonian in terms of these new currents is

$$H_j = \frac{1}{\sqrt{1+\theta^2}} \int dx : \left[j_+^2 + j_-^2 \right] : . \tag{4.14}$$

Using the bosonization rules, we can write the hamiltonian in terms of fermionic fields as

$$H_F = \frac{1}{\sqrt{1+\theta^2}} \int dx : \left[i\psi_-^\dagger \frac{d}{dx}\psi_- - i\psi_+^\dagger \frac{d}{dx}\psi_+ \right] : . \tag{4.15}$$

The fermionic two-point functions for this system can be computed straightforwardly using the bosonization rules (4.1)-(4.2) and (4.7). Indeed, we have

$$\langle \psi_\pm^\dagger(x)\psi_\pm(0)\rangle \sim \frac{1}{(x+vt)^{\frac{\gamma_\pm}{2}}(x-vt)^{\frac{\gamma_\mp}{2}}} , \tag{4.16}$$

where

$$\gamma_\pm = 1 \pm \frac{1}{\sqrt{1+\theta^2}} = 1 \pm \frac{c}{c'} , \tag{4.17}$$

and in the limit $\theta \to 0$, the conventional propagator is correctly recovered. The remaining Green's functions have the forms

$$\langle \psi_\pm^\dagger(x)\psi_\mp(0)\rangle \sim \left(\frac{x+vt}{x-vt}\right)^{\frac{\theta}{2(1+\theta^2)}} , \tag{4.18}$$

which becomes trivial in the commutative limit.

From (4.16), we see that although (4.15) formally describes a 'free' fermion, the Green's function (4.16) reflects an interaction due to non-commutativity in the field space that disappear in the limit $\theta \to 0$. However, even in the non-commutative regime Lorentz invariance remains intact if we scale the speed of light c to c' as defined in (3.7) and as we discuss in the next section. In this sense, the system behaves like a dielectric with a dielectric constant related to the parameter of non-commutativity.

Finally we would like to close this section with some comments about the nature of the conformal field theory and the Virasoro algebra associated with this system. Given the Kac-Moody algebra in (4.11)-(4.12), we can compute the Virasoro algebra in a straightforward manner. Let us define

$$L_n = \frac{1}{\sqrt{1+\theta^2}} \int_{-\pi}^{\pi} dx\, e^{inx} : j^2(x) : . \tag{4.19}$$

It can now be easily seen that

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{\tilde{c}}{12} n(n^2+1)\delta_{n+m} , \tag{4.20}$$

where

$$\tilde{c} = 1 , \tag{4.21}$$

is the central charge of the Virasoro algebra [19]. Thus, we conclude that model of chiral bosons in a non-commutative field space defines a rational conformal field theory with the Kac-Moody level $(1+\theta^2)$ and the central charge of the Virasoro algebra 1.

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5. Relativistic symmetries and Lorentz invariance

In the last section we found that chiral bosons in the non-commutative field space can be mapped to a 'free ' fermion with Green's functions given by (4.16)-(4.18). This result is surprising and unexpected. In addition, the commutator (3.1) explicitly breaks conventional Lorentz invariance and, therefore, the main question is whether there are other generalized Lorentz symmetries in the non-commutative regime.

In appendix B, we discuss how Lorentz transformations are modified if the speed of light is changed as in (3.7). We note here that theories in two-dimensions are very special. Indeed, the parameter θ is dimensionless and, therefore, the only effect produced by non-commutativity is to modify the speed of light. However, this modification is non-trivial because although the only change in Lorentz transformations due to this is to make the replacement (see eq. (3.7))

$$c \to c' = c\sqrt{1+\theta^2}\,, \tag{5.1}$$

the physical interpretation is non-conventional. Namely, the rescaling in c leads us to define the ratio

$$n = \frac{c}{c'} = \frac{1}{\sqrt{1+\theta^2}}\,, \tag{5.2}$$

which can be thought of as a two-dimensional index of refraction which is directly related to the parameter of non-commutativity in the field space. Thus, we can think of a sort of transition between the commutative and the non-commutative regimes where the velocity of the photon, for example, increase when it leaves the commutative regime.

This interpretation, however, cannot be extended to higher dimensions because, in such case, the dimension of θ is nontrivial in units of energy. Consequently, a new non-commutative dimensional parameter with inverse dimensions needs to be introduced for such a phenomenon. As has been discussed in previous papers [9], one can add a new non-commutative parameter in the commutators of momenta in the phase space. If we denote such a parameter as B (in analogy to motion in a magnetic field), then θ and B play the roles of the ultraviolet and infrared cutoffs as discussed in [9]. When the product θB is dimensionless, the limits $\theta B \gg 1$ and $\theta B \ll 1$ define the ultraviolet and infrared regimes respectively. The case $\theta B = 1$ is singular and it would correspond to the critical point suggested above.[4] Thus, in higher dimensions, the phase space considerations are necessary if we want to retain extended relativistic invariance. This can, in fact, be argued in favor of extra dimensions. We would also like to emphasize that our calculations have a direct application to cosmology where some authors have argued that the speed of light could be bigger than c in the first instants of the universe [22].

If the chiral bosons are equivalent to a massless 'free' fermion, then we can imagine the hamiltonian (4.9) to come from

$$\mathcal{L} = \bar{\psi} i \partial\!\!\!/ \psi\,, \tag{5.3}$$

[4]This limit was discussed in [20] and later in many references on non-commutative quantum mechanics [21].

where the temporal component of the derivative in $\partial\!\!\!/$ should be understood as $\partial_0 = \frac{1}{c'}\frac{\partial}{\partial t}$. With this interpretation, the dispersion relations for bosons and fermions hold automatically and can be computed directly from $1/\partial\!\!\!/$ leading to

$$\omega = |E| = c'|k|\,,$$

in full agreement with (3.8).

Thus, relativistic invariance implies that there is no difference in the dispersion relation for bosons and fermions and is determined in terms of the rescaled speed of light (5.1). For a non-relativistic system, on the other hand, c' can simply be thought of as the Fermi velocity with the system discussed above describing a fermion embedded in a medium.

6. Conclusions

In this paper we have discussed the dynamics of chiral bosons defined in a non-commutative field space and have generalized the abelian bosonization to such a case. Our results go over smoothly to the commutative limit and contain the Mandelstam formulae as a special case. As has been discussed in previous papers [9], the modification of the canonical commutation relations is equivalent to adding interactions to the lagrangian (or the hamiltonian) with the standard commutation relations. This is easily seen from the form of the action in (3.10). A constraint analysis of this system leads to the commutators in (3.1) as well as the hamiltonian equations discussed earlier. In spite of the non-commutativity in the field space, the model possesses Lorentz invariance with a scaled speed of light. In this sense, the behavior of the system is reminiscent of a dielectric. The current algebra of the system shows that the system is described by a rational conformal field theory with the Kac-Moody level $(1+\theta^2)$ and the central charge in the Virasoro algebra equal to 1. We would like to point out here that more recently chiral bosons in a non-commutative space-time have been discussed in [23] from a different perspective. However, our approach has been to analyze the model of chiral bosons with non-commutativity in the field space and, consequently, our approach is quite distinct.

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A. Derivation of the Green's functions

In this appendix we will derive the forms of the time ordered Green's function discussed in (3.17) starting from a canonical quantization of the theory. Let us start with the definition of the Feynman Green's function

$$\mathcal{G}_{ab}(x,t) = \theta(t)\langle\Omega|\phi_a(t,x)\phi_b(0)|\Omega\rangle + \theta(-t)\langle\Omega|\phi_b(0)\phi_a(t,x)|\Omega\rangle\,, \tag{A.1}$$

where $\theta(t)$ is the step function and Ω denotes the vacuum of the theory. This defines the time ordered Green's functions of the theory. Using (3.11) as well as (3.1), it is straightforward to check from the definition in (A.1) that

$$\mathcal{D}\mathcal{G}(t,x) = i\Delta(\theta)\delta^2(x)\,, \tag{A.2}$$

where $\mathcal{D}$ is the differential operator given in (3.11), and $\Delta(\theta)$ is defined in (3.16). Comparing with (3.12), we identify (in the matrix notation)

$$\mathcal{G}(x) = iG(x)\Delta(\theta)\,,$$

as pointed out earlier.

To calculate the time ordered Green's function, let us decompose the chiral boson fields in the non-commutative field space in terms of plane waves as

$$\begin{aligned}
\phi_+(x) &= \sqrt{\frac{c'}{c}}\int_0^\infty \frac{dk}{\sqrt{4\pi|k^0|}}\left\{a(k)\,e^{-ik\cdot x} + a^\dagger(k)\,e^{ik\cdot x}\right\},\\
\phi_-(x) &= \sqrt{\frac{c'}{c}}\int_0^\infty \frac{dk}{\sqrt{4\pi|k^0|}}\left\{b^\dagger(k)\,e^{-ik\cdot x} + b(k)\,e^{ik\cdot x}\right\},
\end{aligned}\tag{A.3}$$

where $|k^0| = c'|k|$ and the creation and annihilation operators a, b, $a^\dagger$ and $b^\dagger$ satisfy,

$$\left[a(k), a^\dagger(k')\right] = \left[b(k), b^\dagger(k')\right] = \delta(k-k')\,, \tag{A.4}$$

$$\left[a(k), b(k')\right] = \left[b^\dagger(k), a^\dagger(k')\right] = -\theta\delta(k-k')\,, \tag{A.5}$$

with all other commutators vanishing. This guarantees the commutation relations between the fields in (3.1). The hamiltonian of the theory in (3.3) can now be expressed as,

$$H = c\int_0^\infty dk\,k\left\{a^\dagger(k)\,a(k) + b(k)\,b^\dagger(k)\right\}. \tag{A.6}$$

However, we note that since the operators $(a, a^\dagger)$ do not commute with $(b, b^\dagger)$, they cannot be thought of as raising/lowering energy eigenstates. Consequently, we need to diagonalize the basis.

The diagonalization is fairly straightforward. Defining a new set of creation and annihilation operators as

$$c^\dagger(k) = \sqrt{\frac{c}{2c'}}\left[\gamma_-{}^{1/2}\,a(k) + \gamma_+^{1/2}\,b^\dagger(k)\right] \tag{A.7}$$

$$d(k) = \sqrt{\frac{c}{2c'}}\left[\gamma_+{}^{1/2}\,a(k) - \gamma_-{}^{1/2}\,b^\dagger(k)\right], \tag{A.8}$$

where as defined earlier,

$$\gamma_\pm = 1 \pm \frac{1}{\sqrt{1+\theta^2}} = 1 \pm \frac{c}{c'}\,,$$

it is easy to check that they satisfy

$$\begin{aligned}
\left[c(k), c^{\dagger}(k')\right] &= \left[d(k), d^{\dagger}(k')\right] = \delta(k-k')\,, \\
\left[c(k), d(k')\right] &= \left[c^{\dagger}(k), d^{\dagger}(k)\right] = \left[c(k), d^{\dagger}(k')\right] = 0\,.
\end{aligned} \tag{A.9}$$

In this new basis, the hamiltonian (A.6) takes the form

$$H = c' \int_0^{\infty} dk\, k \left\{ c(k)\, c^{\dagger}(k) + d^{\dagger}(k)\, d(k) \right\}, \tag{A.10}$$

with

$$c' = c\sqrt{1+\theta^2}\,.$$

The field decomposition in this new basis is easily obtained to be

$$\phi_+(t,x) = \sqrt{\frac{c'}{c}} \int_0^{\infty} \frac{dk}{2\sqrt{2\pi k}} \left[\gamma_-\, c^{\dagger}(k)\, e^{-ik(x-c't)} + \gamma_+\, d(k)\, e^{-ik(x+c't)} \right] + \text{c.c.} \tag{A.11}$$

$$\phi_-(t,x) = \sqrt{\frac{c'}{c}} \int_0^{\infty} \frac{dk}{2\sqrt{2\pi k}} \left[\gamma_+\, c^{\dagger}(k)\, e^{-ik(x-c't)} - \gamma_-\, d(k)\, e^{-ik(x+c't)} \right] + \text{c.c.}\,. \tag{A.12}$$

Furthermore, using the integral representation for the step function,

$$\theta(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{e^{i\omega t}}{\omega - i\epsilon}\,,$$

it is a matter of algebra to check that the time ordered Green's function in (A.1) is given by,

$$\begin{aligned}
\mathcal{G}_{++}(t,x) &= -\frac{c'}{8\pi c} \left\{ \gamma_+ \log|x+c't| + \gamma_- \log|x-c't| \right\}, \\
\mathcal{G}_{--}(t,x) &= -\frac{c'}{8\pi c} \left\{ \gamma_- \log|x+c't| + \gamma_+ \log|x-c't| \right\}, \\
\mathcal{G}_{+-}(t,x) &= \mathcal{G}_{-+}(t,x) = -\frac{\theta}{8\pi} \log\left| \frac{x-c't}{x+c't} \right|.
\end{aligned} \tag{A.13}$$

B. Modified Lorentz transformations

Now we discuss the question of relativistic covariance of this theory. In order to do that, let us consider the equation of motion (3.4), i.e.

$$\partial_0 \phi_+ - \partial_1 (\phi_+ - \theta\phi_-) = 0 \tag{B.1}$$

$$\partial_0 \phi_- + \partial_1 (\phi_- + \theta\phi_+) = 0 \tag{B.2}$$

where $x^0 = ct$ and $x^1 = x$.

Now we want to identify a symmetry transformation of the fields related to the linear coordinates transformations given by,

$$x^{0\,\prime} = ax^0 + bx^1 \tag{B.3}$$

$$x^{1\,\prime} = px^0 + qx^1 \tag{B.4}$$

where the coefficients a, b, p and q will be determined below. If we make a coordinate transformations as above, then the derivatives will transform as,

$$\partial_0 \to a\partial_0 + p\partial_1 \tag{B.5}$$
$$\partial_1 \to b\partial_0 + q\partial_1 \tag{B.6}$$

and hence, the equations of motion will take the forms,

$$\partial_0\left[(a-b)\phi_+ + b\theta\phi_-\right] + \partial_1\left[(p-q)\phi_+ + q\theta\phi_-\right] = 0\,, \tag{B.7}$$
$$\partial_0\left[(a+b)\phi_- + b\theta\phi_+\right] + \partial_1\left[(p+q)\phi_- + q\theta\phi_+\right] = 0\,, \tag{B.8}$$

The equations of motions will be invariant under this transformations, if the fields $\phi_\pm$ transform suitably. Indeed, by comparison we find from (B.7), (B.8) and (3.4) that

$$\tilde{\phi}_+ = (a-b)\phi_+ + b\theta\phi_- \tag{B.9}$$
$$\tilde{\phi}_- = (a+b)\phi_- + b\theta\phi_+ \tag{B.10}$$

and, therefore, one have

$$\tilde{\phi}_+ - \theta\tilde{\phi}_- = (q-p)\phi_+ - q\theta\phi_- \tag{B.11}$$
$$\tilde{\phi}_- + \theta\tilde{\phi}_+ = (p+q)\phi_- + q\theta\phi_+\,. \tag{B.12}$$

Multiplying the second equation by θ and adding to the first one, we get

$$\tilde{\phi}_+ = \left[q - p\frac{1}{1+\theta^2}\right]\phi_+ + p\frac{\theta}{1+\theta^2}\phi_-\,.$$

Similarly, by multiplying the first equation by θ and subtracting the second equation, we obtain

$$\tilde{\phi}_- = p\frac{\theta}{1+\theta^2}\phi_+ + \left[q + p\frac{1}{1+\theta^2}\right]\phi_-\,.$$

These equations are compatible with the other expressions for $\tilde{\phi}_\pm$, if $a = q$ and $b = \frac{p}{1+\theta^2}$.

Finally, if we rescale the unit of time such that,

$$x^0 \to \frac{x^0}{\sqrt{1+\theta^2}}\,,$$

the coordinate transformation is orthogonal, i.e.

$$x^{0\,\prime} = ax^0 + \frac{p}{\sqrt{1+\theta^2}}x^1\,, \tag{B.13}$$
$$x^{1\,\prime} = \frac{p}{\sqrt{1+\theta^2}}x^0 + ax^1\,. \tag{B.14}$$

Using the same notation as in the standard Lorentz transformation where $a \equiv \gamma$ and $p \equiv -\gamma\beta$ with $\beta = \frac{v}{c}$ with c the speed of light, it is easy to check that the invariance of the space-time interval implies that,

$$\gamma = \frac{1}{\sqrt{1+\frac{v^2}{c^2(1+\theta^2)}}}\,,$$

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so that the final expression for the symmetry transformations are,

$$x^{0\,\prime} = \frac{x^0 - \frac{v}{c'}x^1}{\sqrt{1-\frac{v^2}{c'^2}}}\,, \tag{B.15}$$

$$x^{1\,\prime} = \frac{x^1 - \frac{v}{c'}x^0}{\sqrt{1-\frac{v^2}{c'^2}}}\,, \tag{B.16}$$

where $c' = c\sqrt{1+\theta^2}$. Namely, the new symmetry transformations are the same as Lorentz transformations with a modified speed of light.

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